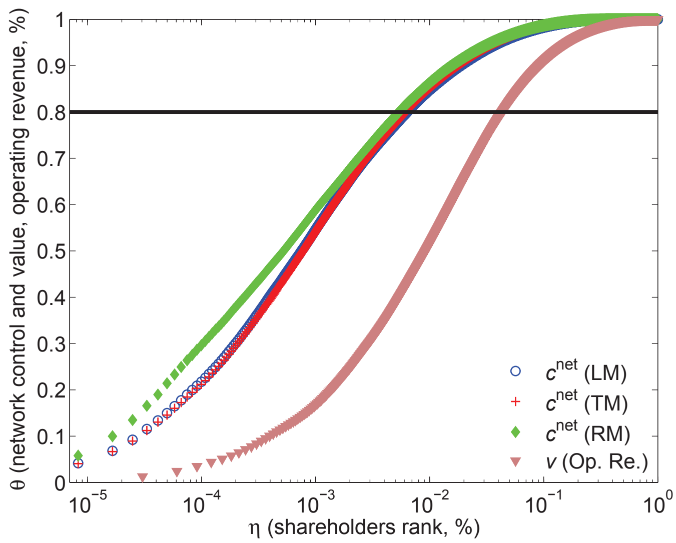

Figure 3: **Concentration of network control and operating revenue.** Economic actors (TNCs and shareholders) are sorted by descending importance, as given by c^{net}. A data point located at (η, θ) corresponds to a fraction η of top economic actors cumulatively holding the fraction θ of network control, value or operating revenue. The different curves refer to network control computed with three models (LM, TM, RM), see SI Appendix, Sec. 3.1, and operating revenue. The horizontal line denotes a value of θ equal to 80%. The level of concentration is determined by the η value of the intersection between each curve and the horizontal line. The scale is semi-log.

This remarkable finding raises at least two questions that are fundamental to the understanding of the functioning of the global economy. Firstly, what are the implication for global financial stability? It is known that financial institutions establish financial contracts, such as lending or credit derivatives, with several other institutions. This allows them to diversify risk, but, at the same time, it also exposes them to contagion [15]. Unfortunately, information on these contracts is usually not disclosed due to strategic reasons. However, in various countries, the existence of such financial ties is correlated with the existence of ownership relations [23]. Thus, in the hypothesis that the structure of the ownership network is a good proxy for that of the financial network, this implies that the global financial network is also very intricate. Recent works have shown that when a financial network is very densely connected it is prone to systemic risk [24, 16]. Indeed, while in good times the network is seemingly robust, in bad times firms go into distress simultaneously. This *knife-edge* property [25, 26] was witnessed during the recent financial turmoil.

Secondly, what are the implications for market competition? Since many TNCs in the core have overlapping domains of activity, the fact that they are connected by ownership relations could facilitate the formation of blocs, which would hamper market competition [14]. Remarkably, the existence of such a core in the global market was never documented before and thus, so far, no scientific study demonstrates or excludes that this international "super-entity" has ever acted as a bloc. However, some examples suggest that this is not an unlikely scenario. For instance, previous studies have shown how even small cross-shareholding structures, at a national level, can affect market competition in sectors such as airline, automobile and steel, as well as the financial one [14, 13]. At the same time, antitrust institutions around the world (e.g., the UK Office of Fair Trade) closely monitor complex ownership structures within their national borders. The fact that international data sets as well as methods to handle large networks became available only very recently, may explain how this finding could go unnoticed for so long.

Two issues are worth being addressed here. One may question the idea of putting together data of ownership across countries with diverse legal settings. However, previous empirical work shows that of all possible determinants affecting ownership relations in different countries (e.g., tax rules, level of corruption, institutional settings, etc.), only the level of investor protection is statistically relevant [27]. In any case, it is remarkable that our results on concentration are robust with respect to three very different models used to infer control from ownership. The second issue concerns the control that financial institutions effectively exert. According to some theoretical arguments, in general, financial institutions do not invest in equity shares in order to exert control. However, there is also empirical evidence of the opposite [23, SI Appendix, Sec. 8.1]. Our results show that, globally, top holders are at least in the position to exert considerable control, either formally (e.g., voting in shareholder and board meetings) or via informal negotiations.

Beyond the relevance of these results for economics and policy making, our methodology can be applied to identify key nodes in any real-world network in which a scalar quantity (e.g., resources or energy) flows along directed weighted links. From an empirical point of view, a bow-tie structure with a very small and influential core is a new observation in the study of complex networks. We conjecture that it may be present in other types of networks where "rich-get-richer" mechanisms are at work (although a degree preferential-attachment [1] alone does not produce a bow-tie). However, the fact that the core is so densely connected could be seen as a generalization of the "rich-club phenomenon" with control in the role of degree [28, 3, SI Appendix, Sec. 8.2]. These related open issues could be possibly understood by introducing control in a "fitness model" [29] of network evolution.

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Supporting Information: The Network of Global Corporate Control

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Contents

1 Acronyms and Abbreviations

The list of acronyms and abbreviations used in the main text and this Supporting Online Material:

BFS: breadth-first search (search algorithm)
CC: (weakly) connected component
FS: financial sector
IN: in-section of a bow-tie
LCC: largest CC
LM: linear model (for estimating control from ownership; see also RM and TM)
NACE: (industry standard classification system)
OCC: other connected components (everything outside the LCC)
OECD: Organization for Economic Co-operation and Development
OR: operating revenue
OUT: out-section of a bow-tie
PC: participated company
RM: relative model (for estimating control from ownership; see also LM and TM)
SCC: strongly connected component (in the main text, this is synonymous with the core of the bow-tie in the LCC)
SH: shareholder (economic actors holding shares in TNCs)
TCH: top control-holder (list of TNCs and SHs that together hold 80% of the network control)
TM: threshold model (for estimating control from ownership; see also LM and RM)
TNC: transnational corporation (OECD definition)
T&T: tubes and tendrils (sections in a bow-tie that either connect IN and OUT, are outgoing from IN, or ingoing to OUT, respectively)

2 Data and TNC Network Detection

The Orbis 2007 marketing database* comprises about 37 million economic actors, both physical persons and firms located in 194 countries, and roughly 13 million directed and weighted ownership links (equity relations). Among many others, information on the industrial classification, geographical position and operating revenue of the actors are provided. This data set is intended to track control relationships rather than patrimonial relationships. Whenever available, the percentage of ownership refers to shares associated with voting rights.

The definition of TNCs given by the OECD[1] states that they

> *[...] comprise companies and other entities established in more than one country and so linked that they may coordinate their operations in various ways, while one or more of these entities may be able to exercise a significant influence over the activities of others, their degree of autonomy within the enterprise may vary widely from one multinational enterprise to another. Ownership may be private, state or mixed.*

Accordingly, we select those companies which hold at least 10% of shares in companies located in more than one country. However, many subsidiaries of large TNCs fulfill themselves this definition of TNCs (e.g. The Coca-Cola Company owns Coca-Cola Hellenic Bottling Company which in turn owns Coca-Cola Beverages Austria). Since for each multinational group we are interested in retaining only one representative, we exclude from the selection the companies for which the so-called ultimate owner (i.e., the owner with the highest share at each degree of ownership upstream of a company*) is quoted in a the stock market. In substitution, we add the quoted ultimate owner to the list (if not already included). In the example above, this procedure identifies only the Coca-Cola Company as a TNC. Overall we obtain a list of 43060 TNCs located in 116 different countries, with 5675 TNCs quoted in stock markets.

Starting from the list of TNCs, we explore recursively the neighborhood of companies in the whole database. First, we proceed downstream of the TNCs (see Fig. S1) with a breadth-first search (BFS) and we identify all companies participated directly and indirectly by the TNCs. We then proceed in a similar way upstream identifying all direct and indirect shareholders of the TNCs. The resulting network can be divided into three classes of nodes, TNC, SH and PC, as shown in Fig. S2. The TNC network constructed in this way consists of 600508 economic entities and 1006987 corporate relations. Notice that it may be possible to reach a PC from several TNCs, or to reach a TNC from several SHs. In other words, paths proceeding downstream or upstream of the TNCs may overlap, giving rise to CCs of various sizes.

It is worthwhile to distinguish the data set constructed here from the one analysed in [5], which was not obtained using a recursive search, but with the simple method of collecting only listed

*URL: `http://www.bvdep.com/en/ORBIS`.

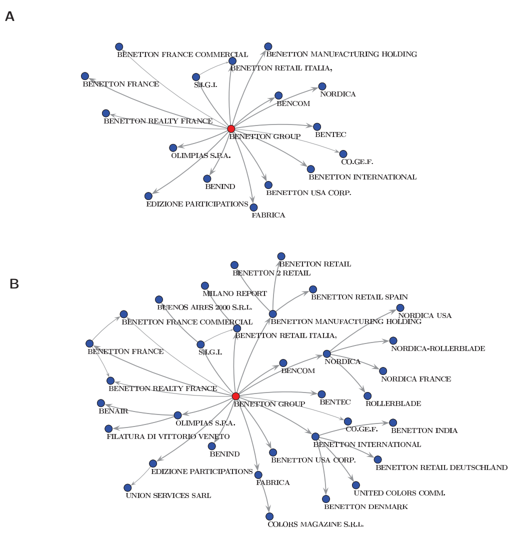

Figure S1: Illustration of the first two steps in the recursive exploration downstream of a TNC. Starting from "Benetton Group" the BFS explores all the direct neighbors (**A**), and then the neighbors' neighbors (**B**).

companies and their direct shareholders. This method neglects all indirect paths involving non-listed companies, so that the true ownership network was only approximated. Moreover, 48 countries were analysed separately, ignoring all cross-country links, an approach which inevitably leaves out entirely the global structure of ownership. The aim there was to construct disjoint national stock market networks, from which the backbones were extracted and analyzed. Here, however, we focus on the entire global topology.

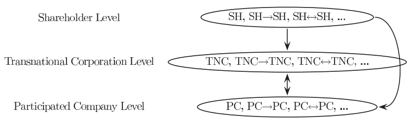

Figure S2: General structure of the TNC network. Three types of economic actors appear: 77456 SHs, 43060 TNCs and 479992 PCs. The network contains in total 600508 nodes, and 1006987 links. Links are mainly from the TNCs to the PCs and amongst the PCs themselves.

3 Network Control

In this section, we first recapitulate in detail the existing method for computing the value or control in a network. In a second step, we highlight two problems that plague this approach, especially in networks with bow-tie topology (see main text, Sec. Network Topology). The first is that the control assigned to firms that are part of cross-shareholding structures is overestimated. The second is a similar overestimation of the control of the shareholders who are themselves not owned by others. These two problems require independent solutions. In particular, the second problem was never raised before in the literature. We provide a novel algorithm that, for the first time, solves both problems and allows the computation of control also for large networks. This method represents a fundamental improvement to previous works, including our own one [5], as explained below in details. Finally, we illustrate the problem and the corrections introduced by the algorithm using a representative example of a small bow-tie network.

3.1 The Existing Methodology

While ownership is an objective quantity given by the percentage of shares owned in a company, control, reflected in voting rights, can only be estimated using a model. There are two steps involved in the derivation of the notion of control we use in this work. Firstly, direct control is estimated from the direct ownership relations. Network control is then computed on the basis of direct control considering all paths in the network.

For the computation of the direct control, we use three models: the linear model, applying the one-share-one-vote rule [2, 3], the threshold model [4] and the relative control model [5]. In the main part of the text, we denote these three models as LM, TM and RM, respectively. According to the LM, there is no deviation between ownership and control, thus the direct control matrix coincides with the ownership matrix, $L_{ij} = W_{ij}$. In the TM, full control over a company is assigned to the actor holding a number of shares higher than a predefined threshold (50% in our case), while the other holders are assigned zero control. The control matrix for the threshold model is denoted as T_{ij}. Finally, the RM assigns control based on the relative fraction of ownership shares that each shareholder has (using a Herfindhal-like concentration index). The control matrix is defined as $R_{ij} := W_{ij}^2/(\sum_{l=1}^{k_j^{in}} W_{lj}^2)$. In particular, the RM assigns high control to a shareholder with a small share in absolute terms, if this share is significantly bigger than the shares of all the other shareholders. For each of these three control matrices, network control is computed with the same procedure. In the main text we use the TM as our main measure, and compare all the results with the LM and the RM. It should be stressed that the global findings are insensitive to the chosen model of direct control.

As explained in the main text, the value of the portfolio of firms owned directly by i should be computed taking into account the value of the firms owned by the firms in the portfolio and so on. Thus, the network portfolio value p_i^{net} consists of the value gained indirectly plus the value of the direct portfolio: $p_i^{\text{net}} = \sum_j W_{ij} v_j + \sum_j W_{ij} p_j^{\text{net}}$. The vector v represents the intrinsic value of the firms (e.g., operating revenue, total assets or market capitalization). Here we use operating revenue, because it is readily available for the economic actors under investigation and it is comparable across sectors (this is not true for total assets). In analogy to the definition above, we introduce the *network control (value)* [5]. This quantity measures the value controlled by a shareholder taking into account the network of firms in which it has direct or indirect shares. In matrix notation,

$$c^{\text{net}} = \mathcal{C} c^{\text{net}} + \mathcal{C} v, \tag{1}$$

where $\mathcal{C} \in \{L, T, R\}$ is one of the three direct control matrices. The solution to Eq. (1) is given by

$$c^{\text{net}} = (I - \mathcal{C})^{-1} \mathcal{C} v =: \tilde{\mathcal{C}} v. \tag{2}$$

For the matrix $(I - \mathcal{C})$ to be non-negative and non-singular, a sufficient condition is that the Frobenius root of $\mathcal{C}$ is smaller than one, $\lambda(\mathcal{C}) < 1$. This is ensured by the following requirement: in each strongly connected component $\mathcal{S}$ there exists at least one node j such that $\sum_{i \in \mathcal{S}} \mathcal{C}_{ij} < 1$. This means that there exists no subset of k firms ($k = 1, \ldots, n$) that are entirely controlled by the k firms themselves, a condition which is always fulfilled.

By taking the series expansion of $(I - \mathcal{C})^{-1}$, it can be proven that: $\mathcal{C}(I - \mathcal{C})^{-1} = (I - \mathcal{C})^{-1}\mathcal{C}$. As a consequence, $\tilde{\mathcal{C}}$ in Eq. (2) coincides with the solution to the equation

$$\tilde{\mathcal{C}}_{ij} = \mathcal{C}_{ij} + \sum_k \tilde{\mathcal{C}}_{ik} \mathcal{C}_{kj}. \tag{3}$$

This is corresponds to the definition of integrated ownership given in [6]. Hence, as in [5], we can interpret c^{net} as the value of control an economic actor gains from all its direct and indirect paths in the network.

Notice that Eq. (1) is related to the notion of eigenvector centrality used to investigate power and influence both in social and economic networks [7, 8]. There is also an additional interpretation of network control in terms a physical system in which a quantity is flowing along the links of the network [5]. In this picture, nodes associated with a value v_j produce v_j units of the quantity at time $t = 1$. The weight of a link ij, given by the adjacency matrix entry A_{ij}, determines the fraction of v_j that flows through it. Then the *inflow*, i.e. the flow ϕ_i entering the node i from each node j at time t is the fraction A_{ij} of the quantity produced by j plus the same fraction of the inflow of j:

$$\phi_i(t+1) = \sum_j A_{ij} \phi_i(t) + \sum_j A_{ij} v_j, \tag{4}$$

In matrix notation, at the steady state, this yields

$$\phi = A\phi + Av, \tag{5}$$

which is formally identical to Eq. (1). Thus if v corresponds to an intrinsic economic value of the nodes, then the network control corresponds to the inflow of control over this value. The network portfolio value of a node is determined by the total inflow of value entering the node.

Next to network control, a related quantity is the so-called *network value*

$$v^{\text{net}} = \mathcal{C} v^{\text{net}} + v, \tag{6}$$

which is akin to a Hubbell index centrality measure [9]. This measure is well-established in the literature [6]. The solution is $v^{\text{net}} = (I - \mathcal{C})^{-1} v$. By noting that

$$\mathcal{C} v^{\text{net}} = \mathcal{C}(I - \mathcal{C})^{-1} v = \tilde{\mathcal{C}} v, \tag{7}$$

we find

$$v^{\text{net}} = \tilde{\mathcal{C}} v + v = c^{\text{net}} + v. \tag{8}$$

In other words, the network value of an economic actor is given by its intrinsic value plus the value gained from network control. It is an estimate of the overall value a corporation has in an ownership network. Notice that network value and network control of a company can differ considerably. As an example, Wall Mart is in top rank by operating revenue but it has no equity shares in other TNCs and thus its network control is zero. In contrast, a small firm can acquire enormous network control via shares in corporations with large operating revenue.

From Eq. (7), where $c^{\text{net}} = \tilde{\mathcal{C}} v = \mathcal{C} v^{\text{net}}$, network control can either be understood as the value of control gained from the intrinsic value reachable by all direct and indirect paths or the value of control given by the network value of directly controlled firms.

3.2 The Algorithm: Computing Control While Remedying the Problems

Unfortunately, the equations defining network control and network value suffer from three drawbacks. Firstly, the computation overestimates control when there are cycles in the network (for example in an SCC[†]), i.e., when the number of inter-firm cross-shareholdings grows [10]. Secondly, as we have discovered, it also leads to paradoxical situations. Consider for instance an SCC that is reachable from a single root-node r that owns an arbitrarily small share in one of the firms in the SCC. The above definition assigns to such a node the sum of the intrinsic value of all the nodes in the SCC. This is obviously not a correct estimate of the control of the node r. These two issues are best understood in the flow analogy. Indeed, in a dense SCC control flows through the nodes many times. The smaller the incoming links from the IN are the longer it takes until the flow stops, as, in the steady state, everything ultimately flows to and accumulates in the root-nodes. However, since control corresponds to the total inflow over an infinite time this exaggerates the control of the nodes in the SCC and all the control ultimately flows to the root-nodes. Thirdly, for large networks, the computation of the inverse matrix can be intractable. Here, for the first time, we overcome the aforementioned problems and propose a new methodology that consists of applying an algorithm to compute network control by treating different components of the network separately.

[†]For more information see SI Sec. 7.

We first illustrate the algorithm for the computation of v^{net}. Then $c^{\text{net}} = v^{\text{net}} - v$. In order to calculate the network value for any specific node i, we extract the whole subnetwork that is downstream of a node i, including i. For this purpose, a breadth-first-search (BFS) returns the set of all nodes reachable from i, going in the direction of the links. Then, all the links among these nodes are obtained from the control matrix of the entire network, except for the links pointing to i which are removed. This ensures that there are no cycles involving i present in the subnetwork. Let $B(i)$ denote the adjacency matrix of such a subnetwork, including i, extracted from the control matrix $\mathcal{C} = (L, T, R)$. Without loss of generality, we can relabel the nodes so that $i = 1$. Since node 1 has now no incoming links, we can decompose $B = B(1)$ as follows:

$$B = \left(\begin{array}{c|c} 0 & d \\ \hline \vec{0} & B^{\text{sub}} \end{array} \right), \tag{9}$$

where d is the row-vector of all links originating from node 1, and B^{sub} is associated with the subgraph of the nodes downstream of i. The value of these nodes is given by the column-vector v^{sub}. By replacing the the matrix B in the expression $v^{\text{net}} = \tilde{\mathcal{C}}v + v = \mathcal{C}(I - \mathcal{C})^{-1}v + v$ and taking the first component we obtain:

$$\begin{aligned} v^{\text{net}}(1) &= \left[B(I-B)^{-1}v\right]_1 + v_1 \\ &= d(I^{\text{sub}} - B^{\text{sub}})^{-1}v^{\text{sub}} + v_1 =: \tilde{d} \cdot v^{\text{sub}} + v_1, \end{aligned} \tag{10}$$

where now $c^{\text{net}}(1) := \tilde{d} \cdot v^{\text{sub}} = d(I^{\text{sub}} - B^{\text{sub}})^{-1}v^{\text{sub}}$.

Notice that if node i has zero in-degree, this procedure yields the same result as the previous formula: $\tilde{B}_{(i,*)} = (0, \tilde{d}) = \tilde{\mathcal{C}}_{(i,*)}$. The notation $A_{(i,*)}$ for a matrix is understood as taking its i-th row. In the next section it is shown that our calculation is in fact equivalent to the correction proposed by [10] to address the problems of the overestimation of network value in the case of ownership due to the presence of cycles.

However, both methods still suffer from the problem of root nodes accumulating all the control. This issue was previously overlooked because the cases analysed did not have a bow-tie structure and because the focus was not on the empirical analysis of control. To solve this issue, we adjust our algorithm to pay special attention to the IN-nodes of an SCC. We partition the bow-tie associated with this SCC into its components: the IN (to which we also add the T&T), the SCC itself, and the OUT. Then, we proceed in multiple steps to compute the network value for all parts in sequence. In this way, the control flows from the OUT, via the SCC to the IN. Finally, the network control is computed from the network value as $c^{\text{net}} = v^{\text{net}} - v$. In detail, our algorithm works as follows:

1. OUT: Compute the network value $v^{\text{net}}(i)$ for all the nodes in the OUT using Eq. (10).

2. OUT $\rightarrow$ SCC: Identify the subset $\mathcal{S}1$ of nodes in the SCC pointing to nodes in the OUT, the latter subset denoted as $\mathcal{O}$. To account for the control entering the SCC from the OUT,

compute the network value of these selected nodes by applying $v^{\text{net}}(s) = \sum_o \mathcal{C}_{so} v^{\text{net}}(o) + v_s$ to them. This is an adaptation of Eq. 8, where s and o are labels of nodes in $\mathcal{S}1$ and $\mathcal{O}$, respectively. Note that we only needed to consider the direct links for this. This computation is also equivalent to applying Eq. (10), which considers the downstream subnetworks of $\mathcal{S}1$, i.e., the whole OUT.

3. SCC: Employ Eq. (10) to the SCC-nodes restricting the BFS to retrieve only nodes in the SCC itself. Note that for those SCC-nodes that were already considered in step 2, their network value is now taken as the intrinsic value in the computation. This means one first needs to assign $v_i \mapsto v^{\text{net}}(i) + v_i$.

4. SCC $\rightarrow$ IN: In this step we solve the problem of the root-nodes acquiring an exaggerated fraction of the network value. For the subset of IN-nodes $\mathcal{I}$ directly connected to some SCC-nodes $\mathcal{S}2$, we again apply $v^{\text{net}}(i) = \sum_s \mathcal{C}_{is} v^{\text{net}}(s) + v_i$, where i and s are labels of nodes in $\mathcal{I}$ and $\mathcal{S}2$, respectively. However, note that due to the cycles present in the SCC, this computation is not equivalent to Eq. (10). In this way only the share of network value over the SCC which is not owned by other SCC-nodes is transferred to the IN-nodes.

5. IN: Finally, use Eq. (10) for assigning the network value to the nodes in the IN-subnetwork. In this case the BFS should not consider the SCC-nodes since their value has been already transfer-ed to their first neighbors in the IN. However, it should retrieve the T&T departing from the IN. Again, for the IN-nodes treated in step 4, first assign $v_i \mapsto v^{\text{net}}(i) + v_i$.

Notice that if any part of the bow-tie structure contains additional smaller SCCs, these should be treated first, by applying steps two to four.

This dissection of the network into its bow-tie components also reduces the computational problems. Although we perform a BFS for each node and compute the inverse of the resulting adjacency matrix of the subnetwork as seen in Eq. (10), the smaller sizes of the subnetworks allow faster computations.

To summarize, using one of the three adjacency matrices estimating direct control, $\mathcal{C} \in \{L, T, R\}$, we can compute the corresponding network value for a corporation: v_i^{net}. By deducting the operating revenue, we retrieve the network control: c_i^{net}. Operating revenue is taken for the value of the TNCs (v_i). Fig. S3 shows the distribution of the operating revenue of the TNCs and the resulting network value.

3.3 Proving the BFS Methodology Corrects for Cycles

Here we show that the BFS algorithm presented in the last section yields an equivalent computation proposed in the literature to address the problems of the presence of cycles leading to exaggerated network value.

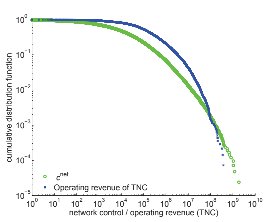

Figure S3: Cumulative distribution function of network control and operating revenue. The network control (TM) in the LCC and the operating revenue of the TNCs in the LCC, from which it is computed, is shown.

In [6] the notion of network value was introduced based on ownership which corresponds, in the case of control, to

$$v^{\text{net}} = \tilde{\mathcal{C}}v + v, \tag{11}$$

which in [10] was identified as being problematic. The authors hence introduced a new model which overcomes this problem of exaggerated indirect value in presence of cycles by introducing

$$\hat{\mathcal{C}}_{ij} := \mathcal{C}_{ij} + \sum_{k \neq i} \hat{\mathcal{C}}_{ik}\mathcal{C}_{kj}. \tag{12}$$

This means that the original matrix $\mathcal{C}$ defined in Eq. (3) is corrected by removing all indirect self-loops of any node i. If the network has no cycles, then Eqs. (3) and (12) yield identical solutions.

We introduce here for the first time a correction operator, that incorporates this modification and makes the associated computations clearer

$$\mathcal{D} := \text{diag}((I - \mathcal{C})^{-1})^{-1} = I - \text{diag}(\hat{\mathcal{C}}), \tag{13}$$

where diag(A) is the matrix of the diagonal of the matrix A. It can be shown that

$$\hat{\mathcal{C}} = \mathcal{D}\tilde{\mathcal{C}}. \tag{14}$$

The associated corrected network value can be identified as

$$\hat{v}^{\text{net}} = \mathcal{D}v^{\text{net}} = \hat{\mathcal{C}}v + \mathcal{D}v. \tag{15}$$

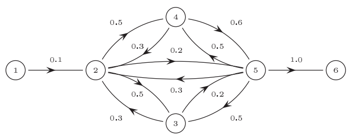

Figure S4: Simple bow-tie network topology. Example with a high degree of interconnectedness of the firms in the strongly connected component (SCC).

Our proposed methodology also corrects for cycles in an equivalent way. This can be seen as follows. By applying the BFS algorithm to node i, we extract the adjacency matrix $B(i)$ of the subnetwork of nodes downstream. From Eq. (12) it holds by construction that

$$\tilde{B}(i)_{ij} = \hat{\mathcal{C}}_{ij} - \hat{\mathcal{C}}_{ii}, \tag{16}$$

where $\tilde{B}(i)$ is defined equivalently to Eq. (2). In a more compact notation

$$\tilde{B}(i)_{i*} = \hat{\mathcal{C}}_{i*} - [\mathrm{diag}(\hat{\mathcal{C}})]_{i*}. \tag{17}$$

Employing Eq. (13) we find that $\tilde{B}(i)_{i*} + I_{i*} = \hat{\mathcal{C}}_{i*} + \mathcal{D}_{i*}$, or equivalently

$$\hat{\mathcal{C}}_{i*}v + \mathcal{D}_{i*}v = \mathcal{D}_{i*}(\tilde{\mathcal{C}}_{i*}v + v_i) = \mathcal{D}_{i*}v^{\text{net}} =: \hat{v}_i^{\text{net}} \tag{18}$$

$$= \tilde{B}_{i*}(i)v + v_i = c^{\text{net}}(i) + v_i =: v^{\text{net}}(i). \tag{19}$$

This concludes that our BFS method and the results in [10] are identical: $\hat{v}_i^{\text{net}} = v^{\text{net}}(i)$.

3.4 An Illustrated Example

Consider the network illustrated in Figure S4. It is an example of a simple bow-tie network topology. The SCC is constructed in a way to highlight the problem of cross-shareholdings. Hence there are many cycles of indirect ownership originating and ending in each firm in the core of the bow-tie.

We assume the underlying value of each firm to be one, i.e., $v = (1, 1, 1, 1, 1, 1)^t$, where t denotes the transposition operation. Moreover, we will employ the TM, hence $\mathcal{C}_ij = W_{ij}$. This results in the network value and the integrated value to be

$$v^{\text{net}} = \begin{pmatrix} 6 \\ 50 \\ 27 \\ 49 \\ 55 \\ 1 \end{pmatrix}, \tag{20}$$

using Eq. (8).

So although the total value present in the network is $6 = \sum_i v_i$, firm 5 has an disproportionately large network control of $v_5^{\text{net}} = 54$, highlighting the problem of overestimating the control in the presence of cycles.

Employing the corrections proposed in [10], i.e. by computing the correction operator defined in Eq. (13), one finds

$$\mathcal{D} = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.162 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.095 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.086 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.000 \end{pmatrix}. \tag{21}$$

From this, the corrected values can be computed from Eq. (15)

$$\hat{v}^{\text{net}} = \begin{pmatrix} 6.000 \\ 5.000 \\ 4.378 \\ 4.667 \\ 4.714 \\ 1.000 \end{pmatrix}. \tag{22}$$

The correction reduces the values of the firms in the core of the bow-tie by approximately one order of magnitude. This confirms that $\hat{v}^{\text{net}}$ and $\hat{c}^{\text{net}}$ are indeed the right measures to consider in the presence of SCCs in the network.

Unfortunately, this example also highlights the second problem of the methodology. It is clear, that root nodes accumulating all the control. As mentioned, our proposed algorithm remedies this problem while still correcting for the overestimation in cycles. One finds from Eq. (10) that

$$\begin{pmatrix} v^{\text{net}}(1) \\ v^{\text{net}}(2) \\ v^{\text{net}}(3) \\ v^{\text{net}}(4) \\ v^{\text{net}}(5) \\ v^{\text{net}}(6) \end{pmatrix} = \begin{pmatrix} 1.500 \\ 5.000 \\ 4.378 \\ 4.667 \\ 4.714 \\ 1.000 \end{pmatrix}, \tag{23}$$

illustrating the change from $v_1^{\text{net}} = \hat{v}_1^{\text{net}} = 6 \geq v^{\text{net}}(1) = 1.5$.

To summarize, employing v^{net} for the computation of control in networks with bow-tie topology overestimates the level of control in the SCC by construction. Using $\hat{v}^{\text{net}}$ on the other hand always assigns the root nodes the highest control. Only the measure $v^{\text{net}}(.)$ puts root and SCC-nodes on par with each other and the leaf-nodes, allowing for the first time an accurate analysis of the control of each node in the network.

3.5 Relations To Previous Work

To summarize, the relation the existing work is as follows. The notion of network value‡ was introduced in [6], in addition to the integrated ownership matrix. This matrix was later corrected in [10].

The notion of network control was first defined in [5] without any of the corrections described above. Because the networks analysed there comprised only listed companies and their direct shareholders, it was sufficient to apply the uncorrected methodology due to the absence of long indirect paths, see SI Sec. 3. In contrast, in the present work, the full-fledged methodology with all the corrections is required in order to consistently compute the flow of control. This resulted in the introduction of the correction operator and its application to the network value and network control. As mentioned, this allowed us to identify a second problem with the methodology. Subsequently, we have incorporated these insights into an algorithm that is suitable for large networks, correcting all potential problems with computing control. Finally, we also uncover the relationship between network control and network value.

‡Although the authors only considered the case of ownership and not that of control, their methods are equivalent to the definition of control employing the LM.